3. Fundamental Properties
3.1 Existence and Uniqueness

For the mathematical model to predict the future state of the system from its current state at $t_0$, the initial-value problem

$$\dot{x} = f(t, x), \quad x(t_0) = x_0 \quad \cdots (3.1)$$

must have a unique solution. This is the question of existence and uniqueness that is addressed in Section 3.1.

By a solution of (3.1) over an interval $[t_0, t_1]$, we mean a continuous function $x: [t_0, t_1] \rightarrow \mathbb{R}^n$ such that $\dot{x}(t)$ is defined and $\dot{x}(t) = f(t, x)$ for all $t \in [t_0, t_1]$. If $f(t, x)$ is continuous in $t$ and $x$, then the solution $x(t)$ will be continuously differentiable.

We will assume that $f(t, x)$ is continuous in $x$, but only piecewise continuous in $t$, in which case, a solution $x(t)$ could only be piecewise continuously differentiable. The assumption that $f(t, x)$ be piecewise continuous in $t$ allows us to include the case when $f(t, x)$ depends on a time-varying input that may experience step changes with time.
3.1 Existence and Uniqueness

A differential equation with a given initial condition might have several solutions. For example, the scalar equation

$$\dot{x} = x^{1/3}, \text{ with } x(0) = 0 \quad \cdots (3.3)$$

has a solution $$x(t) = (2t/3)^{3/2}$$. This solution is not unique, since $$x(t) = 0$$ is another solution. Extra conditions must be imposed on the function $$f$$.

**Theorem 3.1 (Local Existence and Uniqueness)** Let $$f(t,x)$$ be piecewise continuous in $$t$$ and satisfy the Lipschitz condition

$$\|f(t,x) - f(t,y)\| \leq L\|x - y\| \quad \cdots (3.2)$$

$$\forall x, y \in B = \{x \in R^n | \|x - x_0\| \leq r\}, \forall t \in [t_0, t_1]$$. Then, there exists some $$\delta > 0$$ such that the state equation

$$\dot{x} = f(t,x) \text{ with } x(t_0) = x_0$$

has a unique solution over $$[t_0, t_0 + \delta]$$.
3.1 Existence and Uniqueness

Proof: We start by noting that if \( x(t) \) is a solution of
\[
\dot{x} = f(t, x), \quad x(t_0) = x_0 \quad \cdots (C.1)
\]
then, by integration, we have
\[
x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds \quad \cdots (C.2)
\]
Viewing the right-hand side of (C.2) as a mapping of the continuous function \( x: [t_0, t_1] \rightarrow \mathbb{R}^n \), and denoting it by \( (Px)(t) \), we can rewrite (C.2) as
\[
x(t) = (Px)(t) \quad \cdots (C.3)
\]
A solution of (C.3) is a fixed point of the mapping \( P \) that maps \( x \) into \( Px \). Existence of a fixed point of (C.3) can be established by using the contraction mapping theorem. This requires defining a Banach space \( \mathcal{X} \) and a closed set \( S \subset \mathcal{X} \) such that \( P \) maps \( S \) into \( S \) and is a contraction over \( S \). Let
\[
\mathcal{X} = C[t_0, t_0 + \delta], \quad \text{with norm} \quad \|x\|_C = \max_{t \in [t_0, t_0 + \delta]} \|x(t)\|
\]
and
\[
S = \{x \in \mathcal{X} \mid \|x - x_0\|_C \leq r\}
\]
where \( r \) is the radius of the ball \( B \) and \( \delta \) is a positive constant to be chosen.
3.1 Existence and Uniqueness

Proof:

We will restrict the choice of $\delta$ to satisfy $\delta \leq t_1 - t_0$ so that $[t_0, t_0 + \delta] \subset [t_0, t_1]$. To show that it maps $S$ into $S$, write

$$(Px)(t) - x_0 = \int_{t_0}^{t} f(s, x(s))ds = \int_{t_0}^{t} [f(s, x(s)) - f(s, x_0) + f(s, x_0)]ds$$

By piecewise continuity of $f$, we know that $f(t, x_0)$ is bounded on $[t_0, t_1]$. Let

$$h = \max_{t \in [t_0, t_1]} \|f(t, x_0)\|$$

Using the Lipschitz condition (3.2) and the fact that for each $x \in S$,

$$\|x(t) - x_0\| \leq \|x - x_0\|_{c} \leq r, \quad \forall t \in [t_0, t_0 + \delta]$$

we obtain

$$\|(Px)(t) - x_0\| \leq \int_{t_0}^{t} [\|f(s, x(s)) - f(s, x_0)\| + \|f(s, x_0)\|]ds \leq \int_{t_0}^{t} [L\|x(s) - x_0\| + h]ds \leq \int_{t_0}^{t} (Lr + h)ds = (t - t_0)(Lr + h) \leq \delta(Lr + h)$$

Hence, choosing $\delta \leq r/(Lr + h)$ ensures that $P$ maps $S$ into $S$. 
3.1 Existence and Uniqueness

Proof:

To show that $P$ is a contraction mapping over $S$, let $x, y \in S$ and consider

$$
\| (Px)(t) - (Py)(t) \| = \left\| \int_{t_0}^{t} [f(s, x(s)) - f(s, y(s))] ds \right\| \leq \int_{t_0}^{t} \| f(s, x(s)) - f(s, y(s)) \| ds \leq \int_{t_0}^{t} L \| x(s) - y(s) \| ds \\
\leq \int_{t_0}^{t} ds L \| x - y \|_C \leq L\delta \| x - y \|_C
$$

Therefore,

$$
\| Px - Py \|_C \leq L\delta \| x - y \|_C \leq \rho \| x - y \|_C \quad \text{for} \quad \delta \leq \frac{\rho}{L}
$$

Thus, choosing $\rho < 1$ and $\delta \leq \rho / L$ ensures that $P$ is a contraction mapping over $S$. By the contraction mapping theorem, we can conclude that if $\delta$ is chosen to satisfy

$$
\delta \leq \min \left\{ t_1 - t_0, \frac{r}{Lr + h'} \frac{\rho}{L} \right\} \quad \text{for} \quad \rho < 1 \quad \cdots (C.4)
$$

then (C.2) will have a unique solution in $S$. Since we are interested in establishing uniqueness of the solution among all continuous functions $x(t)$, that is, uniqueness in $\chi$. It turns out that any solution of (C.2) in $\chi$ will lie in $S$. 
3.1 Existence and Uniqueness

Proof:
Since $x(t_0) = x_0$ is inside the ball $B$, any continuous solution $x(t)$ must lie inside $B$ for some interval of time. Suppose that $x(t)$ leaves the ball $B$ and let $t_0 + \mu$ be the first time $x(t)$ intersects the boundary of $B$. Then,

$$\|x(t_0 + \mu) - x_0\| = r$$

On the other hand, for all $t \leq t_0 + \mu$

$$\|x(t) - x_0\| \leq \int_{t_0}^{t} \|f(s, x(s)) - f(s, x_0)\| + \|f(s, x_0)\| ds \leq \int_{t_0}^{t} [L\|x(s) - x_0\| + h] ds \leq \int_{t_0}^{t} (Lr + h) ds$$

$$= (t - t_0)(Lr + h) \leq \mu(Lr + h)$$

$$\|x(t_0 + \mu) - x_0\| = r \leq \mu(Lr + h)$$

Therefore,

$$\mu \geq \frac{r}{(Lr + h)} \geq \delta$$

Hence, the solution $x(t)$ cannot leave the set $B$ within the time interval $[t_0, t_0 + \delta]$, which implies that any solution in $\chi$ lies in $S$. Consequently, uniqueness of the solution in $S$ implies uniqueness in $\chi$. 
3.1 Existence and Uniqueness

A function satisfying (3.2) is said to be Lipschitz in \( x \), and the positive constant \( L \) is called a Lipschitz constant.

We also use the words locally Lipschitz and globally Lipschitz to indicate the domain over which the Lipschitz condition holds.

Let us introduce the terminology first for the case when \( f \) depends only on \( x \).

A function \( f(x) \) is said to be locally Lipschitz on a domain (open and connected set) \( D \subset \mathbb{R}^n \) if each point of \( D \) has a neighborhood \( D_0 \) such that \( f \) satisfies the Lipschitz condition (3.2) for all points in \( D_0 \) with some Lipschitz constant \( L_0 \).

We say that \( f \) is Lipschitz on a set \( W \) if it satisfies (3.2) for all points in \( W \), with the same Lipschitz constant \( L \).

Function \( f(x) \) is said to be globally Lipschitz if it is Lipschitz on \( \mathbb{R}^n \).

The same terminology is extended to a function \( f(t,x) \), provided the Lipschitz condition holds uniformly in \( t \) for all \( t \) in a given interval of time.
3.1 Existence and Uniqueness

When \( f: R \to R \), the Lipschitz condition can be written as

\[
\frac{|f(t, x) - f(t, y)|}{|x - y|} \leq L
\]

Therefore, any function \( f(x) \) that has infinite slope at some point is not locally Lipschitz at that point. If \( |f'(x)| \) is bounded by a constant \( k \) over the interval of interest, then \( f(x) \) is Lipschitz on the same interval with Lipschitz constant \( L = k \). This observation extends to vector-valued functions, as demonstrated by Lemma 3.1.

**Lemma 3.1** Let \( f: [a, b] \times D \to R^m \) be, continuous for some domain \( D \subset R^n \). Suppose that \( \frac{\partial f}{\partial x} \) exists and is continuous on \( [a, b] \times D \). If, for a convex subset \( W \subset D \), there is a constant \( L \geq 0 \) such that

\[
\left\| \frac{\partial f}{\partial x} (t, x) \right\| \leq L
\]

on \( [a, b] \times W \), then

\[
\|f(t, x) - f(t, y)\| \leq L\|x - y\|
\]

for all \( t \in [a, b], x \in W, \) and \( y \in W \).
3.1 Existence and Uniqueness

Proof: Let \( \| \cdot \|_p \) be the underlying norm for any \( p \in [1, \infty] \), and determine \( q \in [1, \infty] \) from the relationship 
\[ \frac{1}{p} + \frac{1}{q} = 1. \]
Fix \( t \in [a, b] \), \( x \in W \), and \( y \in W \). Define \( \gamma(s) = (1-s)x + sy \) for all \( s \in \mathbb{R} \) such that \( \gamma(s) \in D \). Since \( W \subset D \) is convex, \( \gamma(s) \in W \) for \( 0 \leq s \leq 1 \). Take \( z \in \mathbb{R}^m \) such that 
\[ \|z\|_q = 1 \text{ and } z^T[f(t,y) - f(t,x)] = \|f(t,y) - f(t,x)\|_p \]
Set \( g(s) = z^T f(t, \gamma(s)) \). Since \( g(s) \) is a real-valued function, which is continuously differentiable in an open interval that includes \([0,1]\), we conclude by the mean value theorem that there is \( s_1 \in (0,1) \) such that
\[ g(1) - g(0) = g'(s_1) \]
Evaluating \( g \) at \( s = 0, s = 1 \), and calculating \( g'(s) \) by using the chain rule, we obtain
\[ g(1) - g(0) = z^T[f(t,y) - f(t,x)] = g'(s_1) = z^T \frac{\partial f}{\partial x}(t, \gamma(s)) \frac{\partial \gamma}{\partial s} = z^T \frac{\partial f}{\partial x}(t, \gamma(s))(y - x) \]
\[ \|f(t,y) - f(t,x)\|_p = z^T[f(t,y) - f(t,x)] = z^T \frac{\partial f}{\partial x}(t, \gamma(s))(y - x) \leq \|z\|_q \left\| \frac{\partial f}{\partial x}(t, \gamma(s)) \right\|_p \|y - x\|_p \leq L\|y - x\|_p \]
where we used the Hölder inequality \( |z^Tw| \leq \|z\|_q \|w\|_p \)
### 3.1 Existence and Uniqueness

**Lemma 3.2** If $f(t, x)$ and $\frac{\partial f}{\partial x}(t, x)$ are continuous on $[a, b] \times D$, for some domain $D \subset \mathbb{R}^n$, then $f$ is locally Lipschitz in $x$ on $[a, b] \times D$.

**Proof:** For $x_0 \in D$, let $r$ be so small that the ball $D_0 = \{x \in \mathbb{R}^n \mid \|x - x_0\| \leq r\}$ is contained in $D$. The set $D_0$ is convex and compact. By continuity, $\frac{\partial f}{\partial x}$ is bounded on $[a, b] \times D$. Let $L_0$ be a bound for $\|\frac{\partial f}{\partial x}\|$ on $[a, b] \times D_0$. By Lemma 3.1, $f$ is Lipschitz on $[a, b] \times D_0$ with Lipschitz constant $L_0$.

**Lemma 3.3** If $f(t, x)$ and $\frac{\partial f}{\partial x}(t, x)$ are continuous on $[a, b] \times \mathbb{R}^n$, then $f$ is globally Lipschitz in $x$ on $[a, b] \times \mathbb{R}^n$ if and only if $\frac{\partial f}{\partial x}$ is uniformly bounded on $[a, b] \times \mathbb{R}^n$. 
Example 3.1  The function \( f(x) = \begin{bmatrix} -x_1 + x_1 x_2 \\ x_2 - x_1 x_2 \end{bmatrix} \)

is continuously differentiable on \( R^2 \). Hence, it is locally Lipschitz on \( R^2 \). It is not globally Lipschitz since \( \frac{\partial f}{\partial x} \) is not uniformly bounded on \( R^2 \). Suppose that we are interested in calculating a Lipschitz constant over the convex set \( W = \{ x \in R^2 \mid |x_1| \leq a_1, |x_2| \leq a_2 \} \). The Jacobian matrix is given by

\[
\begin{bmatrix}
\frac{\partial f}{\partial x} \\
\end{bmatrix} = \begin{bmatrix}
-1 + x_2 & x_1 \\
-x_2 & 1 - x_1 \\
\end{bmatrix}
\]

Using \( \| \cdot \|_\infty \) for vectors in \( R^2 \) and the induced matrix norm for matrices, we have

\[
\left\| \frac{\partial f}{\partial x} \right\|_\infty = \max\{|-1 + x_2| + |x_1|, |x_2| + |1 - x_1|\}
\]

All points in \( W \) satisfy

\[
|-1 + x_2| + |x_1| \leq 1 + a_2 + a_1 \quad \text{and} \quad |x_2| + |1 - x_1| \leq a_2 + 1 + a_1
\]

Hence,

\[
\left\| \frac{\partial f}{\partial x} \right\|_\infty \leq 1 + a_1 + a_2 \quad \text{and a Lipschitz constant can be taken as} \quad L \leq 1 + a_1 + a_2
\]
Example 3.2 The function \( f(x) = -\text{sat}(x_1 + x_2) \)

is not continuously differentiable on \( \mathbb{R}^2 \). Let us check its Lipschitz property by examining \( f(x) - f(y) \). Using \( \|\cdot\|_2 \) for vectors in \( \mathbb{R}^2 \) and the fact that the saturation function \( \text{sat}(\cdot) \) satisfies

\[
|\text{sat}(\eta) - \text{sat}(\xi)| \leq |\eta - \xi|
\]

we obtain

\[
\|f(x) - f(y)\|_2^2 = (x_2 - y_2)^2 + (\text{sat}(x_1 + x_2) - \text{sat}(y_1 + y_2))^2 \leq (x_2 - y_2)^2 + (x_1 + x_2 - y_1 - y_2)^2
\]

\[
= (x_1 - y_1)^2 + 2(x_1 - y_1)(x_2 - y_2) + 2(x_2 - y_2)^2
\]

Using the inequality

\[
a^2 + 2ab + 2b^2 = \begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \leq \lambda_{\max} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \times \|\begin{bmatrix} a \\ b \end{bmatrix}\|_2^2
\]

we conclude that

\[
\|f(x) - f(y)\|_2^2 \leq 2.618\|x - y\|_2^2 \quad \|f(x) - f(y)\|_2 \leq \sqrt{2.618}\|x - y\|_2
\]

A more conservative (larger) Lipschitz constant will be obtained if we use the more conservative inequality

\[
a^2 + 2ab + 2b^2 \leq a^2 + 2ab + 2b^2 \leq 2a^2 + 3b^2 \leq 3(a^2 + b^2)
\]

resulting in a Lipschitz constant \( L = \sqrt{3} \)

\[
2ab \leq 2\sqrt{a^2b^2} \leq a^2 + b^2
\]
3.1 Existence and Uniqueness

Due to equivalence of norms, the choice of a norm on $\mathbb{R}^n$ does not affect the Lipschitz property of a function. Due to equivalence of norms, the choice of a norm on $\mathbb{R}^n$ does not affect the Lipschitz property of a function.

Theorem 3.1 is a local theorem since it guarantees existence and uniqueness only over an interval $[t_0, t_0 + \delta]$, where $\delta$ may be very small. One may try to extend the interval of existence by repeated applications of the local theorem.

However, in general, the interval of existence of the solution cannot be extended indefinitely because the conditions of Theorem 3.1 may cease to hold. There is a maximum interval $[t_0, T)$ where the unique solution starting at $(t_0, x_0)$ exists.

In general, $T$ may be less than $t_1$, in which case as $t_0 \to T$, the solution leaves any compact set over which $f$ is locally Lipschitz in $x$. 
Example 3.3  \( \dot{x} = -x^2 \), with \( x(0) = -1 \)

The function \( f(x) = -x^2 \) is locally Lipschitz for all \( x \in \mathbb{R} \). Hence, it is Lipschitz on any compact subset of \( \mathbb{R} \). The unique solution

\[
x(t) = \frac{1}{t - 1}
\]

exists over \([0,1]\). As \( t \to 1 \), \( x(t) \) leaves any compact set.

\[-x^{-2}dx = dt \quad x^{-1} = t + C \quad C = -1\]

The phrase “finite escape time” is used to describe the phenomenon that a trajectory escapes to infinity at a finite time. In Example 3.3, we say that the trajectory has finite escape time at \( t = 1 \).
3.1 Existence and Uniqueness

**Theorem 3.2 (Global Existence and Uniqueness)** Suppose that \( f(t,x) \) is piecewise continuous in \( t \) and satisfies

\[
\|f(t,x) - f(t,y)\| \leq L\|x - y\| \quad \forall \ x, y \in \mathbb{R}^n, \forall \ t \in [t_0, t_1].
\]

then, the state equation \( \dot{x} = f(t,x) \), with \( x(t_0) = x_0 \), has a unique solution over \([t_0, t_1]\).

**Proof:**

The key point of the proof is to show that the constant \( \delta \) of Theorem 3.1 can be made independent of the initial state \( x_0 \).

From (C.4), we see that the dependence of \( \delta \) on the initial state comes through the constant \( h \) in the term \( \delta \leq r/(Lr + h) \). Since in the current case the Lipschitz condition holds globally, we can choose \( r \) arbitrarily large. Therefore, for any finite \( h \), we can choose \( r \) large enough so that \( r/(Lr + h) > \rho/L \). This reduces (C.4) to the requirement

\[
\delta \leq \min \left\{ t_1 - t_0, \frac{\rho}{L} \right\} \quad \text{for} \quad \rho < 1
\]

If \( t_1 - t_0 \leq \rho/L \), we could choose \( \delta = t_1 - t_0 \) and be done. Otherwise, we choose \( \delta \) to satisfy \( \delta \leq \rho/L \). Now, divide \([t_0, t_1]\) into a finite number of subintervals of length \( \delta \leq \rho/L \), and apply Theorem 3.1 repeatedly.
3.1 Existence and Uniqueness

Example 3.4 Consider the linear system \( \dot{x} = A(t)x + g(t) = f(t, x) \)

where \( A(t) \) and \( g(t) \) are piecewise continuous functions of \( t \). Over any finite interval of time \([t_0, t_1]\), the elements of \( A(t) \) are bounded. Hence, \( \|A(t)\| \leq a \), where \( \|A\| \) is any induced matrix norm. The conditions of Theorem 3.2 are satisfied since

\[
\|f(t, x) - f(t, y)\| = \|A(t)(x - y)\| \leq \|A(t)\||x - y| \leq a|x - y|
\]

for all \( x, y \in \mathbb{R}^n \) and \( t \in [t_0, t_1] \) Therefore, Theorem 3.2 shows that the linear system has a unique solution over \([t_0, t_1]\). Since \( t_1 \) can be arbitrarily large, we can also conclude that if \( A(t) \) and \( g(t) \) are piecewise continuous \( \forall \ t \geq t_0 \), then the system has a unique solution \( \forall \ t \geq t_0 \). Hence, the system cannot have a finite escape time.

The global Lipschitz property is restrictive. Models of many physical systems fail to satisfy it. One can easily construct smooth meaningful examples that do not have the global Lipschitz property, but do have unique global solutions.
3.1 Existence and Uniqueness

Example 3.5  Consider the scalar system \( \dot{x} = -x^3 = f(x) \)

The function \( f(x) \) does not satisfy a global Lipschitz condition since the Jacobian \( \frac{\partial f}{\partial x} = -3x^2 \) is not globally bounded. Nevertheless, for any initial state \( x(t_0) = x_0 \), the equation has the unique solution

\[
x(t) = \text{sign}(x_0) \sqrt{\frac{x_0^2}{1 + 2x_0^2(t - t_0)}}
\]

which is well defined for all \( t \geq t_0 \).

Theorem 3.3  Let \( f(t,x) \) be piecewise continuous in \( t \) and locally Lipschitz in \( x \) for all \( t \geq t_0 \) and all \( x \) in a domain \( D \subset \mathbb{R}^n \). Let \( W \) be a compact subset of \( D \), \( x_0 \in W \), and suppose it is known that every solution of

\[
\dot{x} = f(t,x), \quad x(t_0) = x_0
\]

lies entirely in \( W \). Then, there is a unique solution that is defined for all \( t \geq t_0 \).
3.1 Existence and Uniqueness

Proof:

Recall the discussion on extending solutions, preceding Example 3.3. By Theorem 3.1, there is a unique local solution over \([t_0, t_0 + \delta]\). Let \([t_0, T)\) be its maximal interval of existence. We want to show that \(T = \infty\). Recall (Exercise 3.26) the fact that if \(T\) is finite, then the solution must leave any compact subset. Since the solution never leaves the compact set \(W\), we conclude that \(T = \infty\).

Example 3.6

Consider again the system

\[
\dot{x} = -x^3 = f(x)
\]

of Example 3.5. The function \(f(x)\) is locally Lipschitz on \(R\). If, at any instant of time, \(x(t)\) is positive, the derivative \(\dot{x}(t)\) will be negative. Similarly, if \(x(t)\) is negative, the derivative \(\dot{x}(t)\) will be positive. Therefore, starting from any initial condition \(x(0) = a\), the solution cannot leave the compact set \(\{x \in R| |x| \leq |a|\}\). Thus, we conclude by Theorem 3.3 that the equation has a unique solution for all \(t \geq 0\).
3.2 Continuous Dependence on Initial Conditions and Parameters

For the solution of the state equation (3.1) to be of any interest, it must depend continuously on the initial state $x_0$, the initial time $t_0$, and the right-hand side function $f(t, x)$. Continuous dependence on the initial time to is obvious from the integral expression

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds$$

Let $y(t)$ be a solution of (3.1) that starts at $y(t_0) = y_0$ and is defined on the compact time interval $[t_0, t_1]$. Given $\varepsilon > 0$, there is $\delta > 0$ such that for all $z_0$ in the ball $\{x \in \mathbb{R}^n \|x - y_0\| < \delta\}$, the equation $\dot{x} = f(t, x)$ has a unique solution $z(t)$ defined on $[t_0, t_1]$, with $z(t_0) = z_0$, and satisfies $\|z(t) - y(t)\| < \varepsilon$ for all $t \in [t_0, t_1]$.

Continuous dependence on the right-hand side function $f$ is defined similarly. A more restrictive, but simpler, mathematical representation is to assume that $f$ depends continuously on a set of constant parameters; that is, $f = f(t, x, \lambda)$, where $\lambda \in \mathbb{R}^p$. The constant parameters could represent physical parameters of the system. Let $x(t, \lambda_0)$ be a solution of $\dot{x} = f(t, x, \lambda_0)$ defined on $[t_0, t_1]$, with $x(t_0, \lambda_0) = x_0$. The solution is said to depend continuously on $\lambda$ if for any $\varepsilon > 0$, there is $\delta > 0$ such that for all $\lambda$ in the ball $\{\lambda \in \mathbb{R}^n \|\lambda - \lambda_0\| < \delta\}$, the equation $\dot{x} = f(t, x, \lambda)$ has a unique solution $x(t, \lambda)$ defined on $[t_0, t_1]$, with $x(t_0, \lambda) = x_0$, and satisfies $\|x(t, \lambda) - x(t, \lambda_0)\| < \varepsilon$ for all $t \in [t_0, t_1]$. 
Theorem 3.4 Let \( f(t,x) \) be piecewise continuous in \( t \) and Lipschitz in \( x \) on \( [t_0, t_1] \times W \) with a Lipschitz constant \( L \), where \( W \subset \mathbb{R}^n \) is an open connected set. Let \( y(t) \) and \( z(t) \) be solutions of
\[
\dot{y} = f(t,y), \quad y(t_0) = y_0
\]
and
\[
\dot{z} = f(t,z) + g(t,z), \quad z(t_0) = z_0
\]
such that \( y(t), z(t) \in W \) for all \( t \in [t_0, t_1] \). Suppose that
\[
\|g(t,x)\| \leq \mu, \quad \forall (t,x) \in [t_0, t_1] \times W
\]
for some \( \mu > 0 \). Then,
\[
\|y(t) - z(t)\| \leq \|y_0 - z_0\| \exp[L(t - t_0)] + \frac{\mu}{L} \{\exp[L(t - t_0)] - 1\}
\]
\( \forall t \in [t_0, t_1] \).

Proof: The solutions \( y(t) \) and \( z(t) \) are given by
\[
y(t) = y_0 + \int_{t_0}^{t} f(s,y(s))ds \quad z(t) = z_0 + \int_{t_0}^{t} [f(s,z(s)) + g(s,z(s))]ds
\]
3.2 Continuous Dependence on Initial Conditions and Parameters

Proof: Subtracting the two equations and taking norms yield

\[ \|y(t) - z(t)\| \leq \|y_0 - z_0\| + \int_{t_0}^{t} \|f(s, y(s)) - f(s, z(s))\| ds + \int_{t_0}^{t} \|g(s, z(s))\| ds \]

\[ \leq \gamma + \mu(t - t_0) + \int_{t_0}^{t} L \|y(s) - z(s)\| ds \]

where \( \|y_0 - z_0\| = \gamma \).

\[ \|y(t) - z(t)\| = A(t) \quad \gamma + \mu(t - t_0) = B(t) \quad \int_{t_0}^{t} LA(s) ds = C(t) \]

\[ D(t) = A(t) - B(t) - C(t) \leq 0 \]

\[ \dot{C}(t) = LA(t) = LC(t) + LB(t) + LD(t) \]

\[ C(t) = U(t)V(t) \]

\[ \dot{C}(t) = U(t) \frac{dV(t)}{dt} + V(t) \frac{dU(t)}{dt} = LU(t)V(t) + LB(t) + LD(t) \]
3.2 Continuous Dependence on Initial Conditions and Parameters

Proof:
\[
\dot{C}(t) = U(t)\frac{dV(t)}{dt} + V(t)\frac{dU(t)}{dt} = LU(t)V(t) + LB(t) + LD(t)
\]

\[
V(t)\frac{dU(t)}{dt} = LU(t)V(t)
\]

\[
\frac{dU(t)}{dt} = LU(t)
\]

\[
U(t) = U(t_0)\exp[L(t - t_0)]
\]

\[
U(t)\frac{dV(t)}{dt} = LB(t) + LD(t)
\]

\[
V(t) = \int_{t_0}^{t} \frac{1}{U(t_0)}\exp[L(t_0 - s)] [LB(s) + LD(s)]ds + V(t_0)
\]

\[
C(t) = U(t)V(t) = \int_{t_0}^{t} \exp[L(t - s)] [LB(s) + LD(s)]ds \leq \int_{t_0}^{t} LB(s)\exp[L(t - s)] ds
\]

\[
A(t) \leq B(t) + C(t) \leq B(t) + \int_{t_0}^{t} LB(s)\exp[L(t - s)] ds
\]

\[
\|y(t) - z(t)\| \leq \gamma + \mu(t - t_0) + \int_{t_0}^{t} L[\gamma + \mu(s - t_0)]\exp[L(t - s)] ds
\]
3.2 Continuous Dependence on Initial Conditions and Parameters

Proof:

\[ \|y(t) - z(t)\| \leq \gamma + \mu(t - t_0) + \int_{t_0}^{t} L[\gamma + \mu(s - t_0)] \exp[L(t - s)] \, ds \]

Integrating the right-hand side by parts, we obtain

\[ \gamma + \mu(t - t_0) + \int_{t_0}^{t} L[\gamma + \mu(s - t_0)] \exp[L(t - s)] \, ds \]

\[ = \gamma + \mu(t - t_0) + [-[\gamma + \mu(s - t_0)] \exp[L(t - s)]]_{t_0}^{t} + \int_{t_0}^{t} \mu \exp[L(t - s)] \, ds \]

\[ = \gamma + \mu(t - t_0) - [\gamma + \mu(t - t_0)] + \gamma \exp[L(t - t_0)] + \frac{\mu}{L} \{\exp[L(t - t_0)] - 1\} \]

\[ = \gamma \exp[L(t - t_0)] + \frac{\mu}{L} \{\exp[L(t - t_0)] - 1\} = \|y_0 - z_0\| \exp[L(t - t_0)] + \frac{\mu}{L} \{\exp[L(t - t_0)] - 1\} \]

\[ \|y(t) - z(t)\| \leq \|y_0 - z_0\| \exp[L(t - t_0)] + \frac{\mu}{L} \{\exp[L(t - t_0)] - 1\} \]
3.2 Continuous Dependence on Initial Conditions and Parameters

**Theorem 3.5** Let \( f(t, x, \lambda) \) be continuous in \((t, x, \lambda)\) and locally Lipschitz in \(x\) (uniformly in \(t\) and \(\lambda\)) on \([t_0, t_1] \times D \times \{\|\lambda - \lambda_0\| \leq c\}\), where \(D \subset \mathbb{R}^n\) is an open connected set. Let \( y(t, \lambda_0) \) be a solution of \( \dot{x} = f(t, x, \lambda_0) \) with \( y(t_0, \lambda_0) = y_0 \in D \). Suppose \( y(t, \lambda_0) \) is defined and belongs to \(D\) for all \(t \in [t_0, t_1]\). Then, given \(\varepsilon > 0\), there is \(\delta > 0\) such that if

\[
\|z_0 - y_0\| < \delta \quad \text{and} \quad \|\lambda - \lambda_0\| < \delta
\]

then there is a unique solution \( z(t, \lambda) \) of \( \dot{x} = f(t, x, \lambda) \) defined on \([t_0, t_1]\), with \( z(t_0, \lambda) = z_0 \) and \( z(t, \lambda) \) satisfies

\[
\|z(t, \lambda) - y(t, \lambda_0)\| < \varepsilon, \quad \forall t \in [t_0, t_1]
\]

**Proof:**

By continuity of \( y(t, \lambda_0) \) in \(t\) and the compactness of \([t_0, t_1]\), we know that \( y(t, \lambda_0) \) is bounded on \([t_0, t_1]\). Define a “tube” \( U \) around the solution \( y(t, \lambda_0) \) (see Figure 3.1) by

\[
U = \{(x, t) \in [t_0, t_1] \times \mathbb{R}^n \mid ||x - y(t, \lambda_0)|| \leq \varepsilon\}
\]

![Figure 3.1: A tube constructed around the solution \( y(t, \lambda_0) \).](attachment:image.png)
3.2 Continuous Dependence on Initial Conditions and Parameters

Proof:

Suppose that $U \subset [t_0, t_1] \times D$; if not, replace $\varepsilon$ by $\varepsilon_1 < \varepsilon$ that is small enough to ensure that $U \subset [t_0, t_1] \times D$ and continue the proof with $\varepsilon_1$. The set $U$ is compact; hence, $f(t, x, \lambda)$ is Lipschitz in $x$ on $U$ with a Lipschitz constant, say, $L$. By continuity of $f$ in $\lambda$, for any $\alpha > 0$, there is $\beta > 0$ (with $(\beta < c)$ such that

$$\|f(t, x, \lambda) - f(t, x, \lambda_0)\| < \alpha, \quad \forall (x, t) \in U, \quad \forall \|\lambda - \lambda_0\| < \beta$$

Take $\alpha < \varepsilon$ and $\|z_0 - y_0\| < \alpha$. By the local existence and uniqueness theorem, there is a unique solution $z(t, \lambda)$ on some time interval $[t_0, t_0 + \Delta]$. We let $\tau$ be the first time the solution leaves the tube and show that we can make $\tau > t_1$. On the time interval $[t_0, \tau]$, the conditions of Theorem 3.4 are satisfied with $\mu = \alpha$. Hence,

$$\|z(t, \lambda) - y(t, \lambda_0)\| < \alpha \exp[\alpha(t - t_0)] + \frac{\alpha}{L} \{\exp[L(t - t_0)] - 1\} < \alpha \frac{L + 1}{L} \exp[L(t - t_0)]$$

Choosing $\alpha < \varepsilon L \exp[-L(t_1 - t_0)]/(L + 1)$ ensures that the solution $z(t, \lambda)$ cannot leave the tube during the interval $[t_0, t_1]$. Therefore, $z(t, \lambda)$ is defined on $[t_0, t_1]$ and satisfies $\|z(t, \lambda) - y(t, \lambda_0)\| < \varepsilon$. Taking $\delta = \min\{\alpha, \beta\}$ completes the proof of the theorem.
Suppose that $f(t, x, \lambda)$ is continuous in $(t, x, \lambda)$ and has continuous first partial derivatives with respect to $x$ and $\lambda$ for all $(t, x, \lambda) \in [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^p$. Let $\lambda_0$ be a nominal value of $\lambda$, and suppose that the nominal state equation

$$\dot{x} = f(t, x, \lambda_0), \quad \text{with } x(t_0) = x_0$$

has a unique solution $x(t, \lambda_0)$ over $[t_0, t_1]$. From Theorem 3.5, we know that for all $\lambda$ sufficiently close to $\lambda_0$, that is, $\|\lambda - \lambda_0\|$ sufficiently small, the state equation

$$\dot{x} = f(t, x, \lambda), \quad \text{with } x(t_0) = x_0$$

has a unique solution $x(t, \lambda)$ over $[t_0, t_1]$ that is close to the nominal solution $x(t, \lambda_0)$.

The continuous differentiability of $f$ with respect to $x$ and $\lambda$ implies the additional property that the solution $x(t, \lambda)$ is differentiable with respect to $\lambda$ near $\lambda_0$. To see this, write

$$x(t, \lambda) = x_0 + \int_{t_0}^{t} f(s, x(s, \lambda), \lambda) \, ds$$
3.3 Differentiability of Solutions and Sensitivity Equations

Taking partial derivatives with respect to $\lambda$ yields

$$x_\lambda(t, \lambda) = \int_{t_0}^{t} \left[ \frac{\partial f}{\partial x}(s, x(s, \lambda), \lambda)x_\lambda(s, \lambda) + \frac{\partial f}{\partial \lambda}(s, x(s, \lambda), \lambda) \right] ds$$

where $x_\lambda(t, \lambda) = [\partial x(t, \lambda)/\partial \lambda]$ and $[\partial x_0/\partial \lambda] = 0$.

Differentiating with respect to $t$, it can be seen that $x_\lambda(t, \lambda)$ satisfies the differential equation

$$\frac{\partial}{\partial t}x_\lambda(t, \lambda) = A(t, \lambda)x_\lambda(t, \lambda) + B(t, \lambda), \quad x_\lambda(t_0, \lambda) = 0 \quad \cdots (3.4)$$

where

$$A(t, \lambda) = \frac{\partial f(t, x, \lambda)}{\partial x} \bigg|_{x=x(t,\lambda)} , \quad B(t, \lambda) = \frac{\partial f(t, x, \lambda)}{\partial \lambda} \bigg|_{x=x(t,\lambda)}$$

For $\lambda$ sufficiently close to $\lambda_0$, the matrices $A(t, \lambda)$ and $B(t, \lambda)$ are defined on $[t_0, t_1]$. Hence, $x_\lambda(t, \lambda)$ is defined on the same interval.
3.3 Differentiability of Solutions and Sensitivity Equations

Let $S(t) = x_\lambda(t, \lambda_0)$; then $S(t)$ is the unique solution of the equation

$$\dot{S}(t) = A(t, \lambda_0)S(t) + B(t, \lambda_0), \quad S(t_0) = 0 \quad \cdots (3.5)$$

The function $S(t)$ is called the sensitivity function, and (3.5) is called the sensitivity equation.

For small $\|\lambda - \lambda_0\|$, $x(t, \lambda)$ can be expanded in a Taylor series about the nominal solution $x(t, \lambda_0)$ to obtain

$$x(t, \lambda) = x(t, \lambda_0) + S(t)(\lambda - \lambda_0) + H. O. T$$

$$x(t, \lambda) \approx x(t, \lambda_0) + S(t)(\lambda - \lambda_0) \quad \cdots (3.6)$$

Knowledge of the nominal solution and the sensitivity function suffices to approximate the solution for all values of $\lambda$ in a (small) ball centered at $\lambda_0$. 
3.3 Differentiability of Solutions and Sensitivity Equations

The procedure for calculating the sensitivity function $S(t)$ is summarized by the following steps:

- Solve the nominal state equation for the nominal solution $x(t, \lambda_0)$.
- Evaluate the Jacobian matrices
  
  
  
  
  
  - Solve the sensitivity equation (3.5) for $S(t)$.

An alternative approach for calculating $S(t)$ is to solve for the nominal solution and the sensitivity function simultaneously. This can be done by appending the variational equation (3.4) with the original state equation, then setting $\lambda = \lambda_0$ to obtain the $(n + np)$ augmented equation

\[
\begin{align*}
\dot{x} &= f(t, x, \lambda_0) \\
\dot{S} &= \left[ \frac{\partial f(t, x, \lambda)}{\partial x} \right]_{\lambda=\lambda_0} S + \left[ \frac{\partial f(t, x, \lambda)}{\partial \lambda} \right]_{\lambda=\lambda_0} \\

\end{align*}
\]

\[x(t_0) = x_0, \hspace{1cm} S(t_0) = 0\]

which is solved numerically.
Example 3.7 Consider the phase-locked-loop model

\[
\begin{align*}
\dot{x}_1 &= x_2 = f_1(x_1, x_2) \\
\dot{x}_2 &= -c \sin x_1 - (a + b \cos x_1)x_2 = f_2(x_1, x_2)
\end{align*}
\]

and suppose the parameters \(a, b,\) and \(c\) have the nominal values \(a_0 = 1, b_0 = 0,\) and \(c_0 = 1.\) The nominal system is given by

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\sin x_1 - x_2
\end{align*}
\]

The Jacobian matrices \([\partial f / \partial x]\) and \([\partial f / \partial \lambda]\) are given by

\[
\frac{\partial f}{\partial x} = \begin{bmatrix}
0 \\
-c \cos x_1 + bx_2 \sin x_1 & -(a + b \cos x_1)
\end{bmatrix}
\]

\[
\frac{\partial f}{\partial \lambda} = \begin{bmatrix}
\frac{\partial f}{\partial a} & \frac{\partial f}{\partial b} & \frac{\partial f}{\partial c}
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 \\
-x_2 & -x_2 \cos x_1 & -\sin x_1
\end{bmatrix}
\]
Example 3.7

Evaluate these Jacobian matrices at the nominal parameters $a = 1$, $b = 0$, and $c = 1$ and to obtain

$$
\frac{\partial f}{\partial x}|_{\text{nominal}} = \begin{bmatrix}
0 & 1 \\
- \cos x_1 & -1
\end{bmatrix}
$$

$$
\frac{\partial f}{\partial \lambda}|_{\text{nominal}} = \begin{bmatrix}
0 & 0 & 0 \\
-x_2 & -x_2 \cos x_1 & -\sin x_1
\end{bmatrix}
$$

Let

$$
S = \begin{bmatrix}
x_3 & x_5 & x_7 \\
x_4 & x_6 & x_8
\end{bmatrix}
$$

Then (3.7) is given by

$$
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\sin x_1 - x_2 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -x_3 \cos x_1 - x_4 - x_2 \\
\dot{x}_5 &= x_6 \\
\dot{x}_6 &= -x_5 \cos x_1 - x_6 - x_2 \cos x_1 \\
\dot{x}_7 &= x_8 \\
\dot{x}_8 &= -x_7 \cos x_1 - x_8 - \sin x_1
\end{align*}
$$

\begin{align*}
x_1(0) &= x_{10} \\
x_2(0) &= x_{20} \\
x_3(0) &= 0 \\
x_4(0) &= 0 \\
x_5(0) &= 0 \\
x_6(0) &= 0 \\
x_7(0) &= 0 \\
x_8(0) &= 0
\end{align*}
3.4 Comparison Principle

Quite often when we study the state equation $\dot{x} = f(t, x)$ we need to compute bounds on the solution $x(t)$ without computing the solution itself. The Gronwall-Bellman inequality is one tool that can be used toward that goal. Another tool is the comparison lemma.

**Lemma 3.4 (Comparison Lemma)** Consider the scalar differential equation

$\dot{u} = f(t, u), \quad u(t_0) = u_0$

where $f(t, u)$ is continuous in $t$ and locally Lipschitz in $u$, for all $t \geq 0$ and all $u \in J \subset \mathbb{R}$. Let $[t_0, T)$ ($T$ could be infinity) be the maximal interval of existence of the solution $u(t)$, and suppose $u(t) \in J$ for all $t \in [t_0, T)$. Let $v(t)$ be a continuous function whose upper right-hand derivative $D^+ v(t)$ satisfies the differential inequality

$D^+ v(t) \leq f(t, v(t)), \quad v(t_0) \leq u_0$

with $v(t) \in J$ for all $t \in [t_0, T)$. Then, $v(t) \leq u(t)$ for all $t \in [t_0, T)$. 
3.4 Comparison Principle

Proof:

The upper right-hand derivative $D^+ v(t)$ is defined by

$$D^+ v(t) = \lim_{h \to 0^+} \sup \frac{v(t + h) - v(t)}{h}$$

- If $v(t)$ is differentiable at $t$, then $D^+ v(t) = \dot{v}(t)$

- If $\frac{1}{h} |v(t + h) - v(t)| \leq g(t, h), \quad \forall \, h \in (0, b]$ and $\lim_{h \to 0^+} g(t, h) = g_0(t)$ then $D^+ v(t) \leq g_0(t)$

To prove lemma 3.4, consider the differential equation

$$\dot{z} = f(t, z) + \lambda, \quad z(t_0) = u_0 \quad \cdots (C.5)$$

where $\lambda$ is a positive constant. On any compact interval $[t_0, t_1]$ we conclude from Theorem 3.5 that for any $\varepsilon > 0$, there is $\delta > 0$ such that if $\lambda < \delta$ then (C.5) has a unique solution $z(t, \lambda)$ defined on $[t_0, t_1]$ and

$$|z(t, \lambda) - u(t)| < \varepsilon, \quad \forall \, t \in [t_0, t_1] \quad \cdots (C.6)$$
3.4 Comparison Principle

Proof:

- Claim 1: \( v(t) \leq z(t, \lambda) \) for all \( t \in [t_0, t_1] \).

This claim can be shown by contradiction, for if it were not true, there would be times \( a, b \in (t_0, t_1) \) such that \( v(a) = z(a, \lambda) \) and \( v(t) > z(t, \lambda) \) for \( a < t \leq b \). Consequently,

\[
v(t) - v(a) > z(t, \lambda) - z(a, \lambda), \quad \forall t \in (a, b]
\]

which implies

\[
D^+ v(t) \geq \dot{z}(a, \lambda) = f(t, z(a, \lambda)) + \lambda > f(a, v(a))
\]

which contradicts the inequality \( D^+ v(t) \leq f(t, v(t)) \).

- Claim 2: \( v(t) \leq u(t) \) for all \( t \in [t_0, t_1] \).

Again, this claim can be shown by contradiction, for if it were not true, there would exist \( a \in (t_0, t_1) \) such that \( v(a) > u(a) \). Taking \( \varepsilon = [v(a) - u(a)]/2 \) and using (C.6), we obtain

\[
v(a) - z(a, \lambda) = v(a) - u(a) + u(a) - z(a, \lambda) \geq \varepsilon
\]

which contradicts the statement of Claim 1. Since this is true on every compact interval, we conclude that it holds for all \( t \in [t_0, T] \).
3.4 Comparison Principle

Example 3.8  The scalar differential equation \( \dot{x} = f(x) = -(1 + x^2)x, \quad x(0) = a \)

has a unique solution on \([0, t_1)\), for some \(t_1 > 0\), because \(f(x)\) is locally Lipschitz. Let \(v(t) = x^2(t)\). The function \(v(t)\) is differentiable and its derivative is given by

\[
\dot{v}(t) = 2x(t)\dot{x}(t) = -2x^2(t) - 2x^4(t) \leq -2x^2(t) = -2v(t)
\]

Hence, \(v(t)\) satisfies the differential inequality

\[
\dot{v}(t) \leq -2v(t), \quad v(0) = a^2
\]

Let \(u(t)\) be the solution of the differential equation

\[
\dot{u} = -2u, \quad u(0) = a^2 \quad \Rightarrow \quad u(t) = a^2 e^{-2t}
\]

Then, by comparison lemma, the solution \(x(t)\) is defined for all \(t \geq 0\) and satisfies

\[
|x(t)| = \sqrt{v(t)} \leq e^{-t}|a|, \quad \forall t \geq 0
\]
Example 3.9  The scalar differential equation
\[ \dot{x} = f(t, x) = -(1 + x^2)x + e^t, \quad x(0) = a \]
has a unique solution on \([0, t_1)\) for some \(t_1 > 0\), because \(f(t, x)\) is locally Lipschitz in \(x\). We want to find an upper bound on \(|x(t)|\) similar to the one we obtained in the previous example. Let us start with \(v(t) = x^2(t)\) as in Example 3.8. The derivative of \(v\) is given by
\[ \dot{v}(t) = 2x(t)\dot{x}(t) = -2x^2(t) - 2x^4(t) + 2x(t)e^t \leq -2v(t) + 2\sqrt{v(t)}e^t \]
The resulting differential equation will not be easy to solve. Instead, we consider a different choice of \(v(t) = |x(t)|\). For \(x(t) \neq 0\), the function \(v(t)\) is differentiable and its derivative is given by
\[ \dot{v}(t) = \frac{d}{dt}\sqrt{x^2(t)} = -\frac{x(t)\dot{x}(t)}{|x(t)|} = -|x(t)||1 + x^2(t)| + \frac{x(t)}{|x(t)|}e^t \leq -|x(t)| + e^t = -v(t) + e^t \]
On the other hand, when \(x(t) = 0\), we have
\[ \frac{|v(t + h) - v(t)|}{h} = \frac{|x(t + h)| - |x(t)|}{h} = \frac{|x(t + h)|}{h} = \frac{1}{h} \left| \int_{t}^{t+h} f(\tau, x(\tau))d\tau \right| = \left| f(t, 0) + \frac{1}{h} \int_{t}^{t+h} [f(\tau, x(\tau)) - f(t, x(t))]d\tau \right| \leq |f(t, 0)| + \frac{1}{h} \int_{t}^{t+h} |f(\tau, x(\tau)) - f(t, x(t))|d\tau \]
3.4 Comparison Principle

Example 3.9 \[ \dot{x} = f(t, x) = -(1 + x^2)x + e^t, \quad x(0) = a \]

\[
\frac{|v(t + h) - v(t)|}{h} \leq |f(t, 0)| + \frac{1}{h} \int_t^{t+h} |f(\tau, x(\tau)) - f(t, x(t))| \, d\tau
\]

Since \( f(t, x(t)) \) is a continuous function of \( t \), given any \( \varepsilon > 0 \) there is \( \delta > 0 \) such that for all \( |\tau - t| < \delta \), \( |f(\tau, x(\tau)) - f(t, x(t))| < \varepsilon \). Hence, for all \( h < \delta \),

\[
\lim_{h \to 0+} \frac{1}{h} \int_t^{t+h} |f(\tau, x(\tau)) - f(t, x(t))| \, d\tau = 0
\]

which shows that

\[
\frac{1}{h} \int_t^{t+h} |f(\tau, x(\tau)) - f(t, x(t))| \, d\tau < \varepsilon
\]

Thus, \( D^+ v(t) \leq |f(t, 0)| = e^t \) whenever \( x(t) = 0 \).
3.4 Comparison Principle

Example 3.9 \( \dot{x} = f(t, x) = -(1 + x^2)x + e^t \), \( x(0) = a \)

\[
D^+ v(t) \leq -v(t) + e^t, \quad v(0) = |a|
\]

Letting \( u(t) \) be the solution of the linear differential equation

\[
\dot{u} = -u + e^t, \quad u(0) = |a|
\]

\[
sU(s) - u(0) = -U(s) + \frac{1}{s-1}
\]

\[
U(s) = \frac{u(0)}{s+1} + \frac{1}{s-1} \cdot \frac{1}{s+1} = \frac{|a|}{s+1} + \frac{1}{2s-1} - \frac{1}{2s+1}
\]

we conclude by the comparison lemma that

\[
v(t) \leq u(t) = |a|e^{-t} + \frac{1}{2}e^{t} - \frac{1}{2}e^{-t}, \quad \forall \, t \in [0, t_1)
\]

The upper bound on \( v(t) \) is finite for every finite \( t_1 \) and approaches infinity only as \( t_1 \to \infty \). Therefore, the solution \( x(t) \) is defined for all \( t > 0 \) and satisfies

\[
|x(t)| \leq |a|e^{-t} + \frac{1}{2}e^{t} - \frac{1}{2}e^{-t}, \quad \forall \, t \geq 0
\]